UNIT - I ALGEBRA

Chapter 1.1 MATRICES AND DETERMINANTS

Introduction:

Matrix and its applications are very important part of Mathematics. Also it is one of the most powerful tools in Mathematics.

Matrix notation and operations are used in Electronic spread sheet programs on personal computer which are used in like business, budgeting, sales projection, cost estimation, analysing the results of an experiment etc. Also many physical operations such as magnification, rotation and reflection through a plane can be represented mathematically by matrices. Also matrices are used in Cryptography.

Matrix:

A Matrix is represented by a rectangular array of numbers (or) functions arranged in rows and columns, put within a bracket.

The numbers (or) functions which are entries in the matrix are called as the element of the matrix.

Examples of Matrices:

Raju has 10 notebooks and 15 pens. Mani has 5 notebooks and 2 pens. Malar has 9 note books and 5 pens.

The above information may be represented in the form of matrix as follows.

$$\begin{array}{l c} & \begin{array}{cc}\text{Note Books} & \text{Pens}\end{array} \\ \begin{array}{l}\text{Raju}\\ \text{Mani}\\ \text{Malar}\end{array} & \begin{bmatrix} 10 & 15 \\ 5 & 2 \\ 9 & 5 \end{bmatrix} \end{array}$$

Consider the linear equations with 3 unknowns a, b, c.

$$a + b + c = 3$$

$$2a - b + c = 2 \text{ and}$$

$$3a + 2b - 2c = 3$$

The above equations can be represented in the form of matrix A by writing the co-efficients of a, b, c in the order which they occur and enclose them within a bracket.

Then we get

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 3 & 2 & -2 \end{bmatrix}$$

Here A is Matrix

The horizontal lines of elements are called as Row of the matrix.

i.e. $\underrightarrow{[\,1\ \ 1\ \ 1\,]}$ I Row of the matrix A

The vertical lines of elements are called column of the matrix.

i.e. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \downarrow$ I Column

Order of a matrix:

If a matrix has m rows and n columns then the order of the matrix is m x n (read as m by n)

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Here A has 3 rows and 2 columns. So the order of matrix A is 3 x 2.

Problems:

Find the order of the following matrix.

(i) $A = \begin{bmatrix} 1 & 3 & -1 \\ 5 & 0 & 2 \\ 7 & 5 & 8 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

(iii) $C = \begin{bmatrix} 7 & 0 & 2 \end{bmatrix}$

(iv) $D = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$

Types of Matrices:

1. **Row Matrix:** A matrix is said to be a row matrix, if it has only one row and any number of columns.

   e.g, $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ is a row matrix of order 1 x 3.

2. **Column Matrix:**

   A matrix is said to be a column matrix, if it has only one column and any number of rows.

   e.g., $B = \begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix}$ is a column matrix of order 3 x 1.

3. **Null (or) Zero Matrix:**

If all the elements of a matrix are zero, then it is called a Null (or) Zero matrix. It is denoted by O.

Eg. $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is zero matrix of order 2 x 2.

4. **Square Matrix:**

In a matrix, if the number of rows and the number of columns of a matrix are equal then the matrix is called a square matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Here, number of rows = number of columns = 3

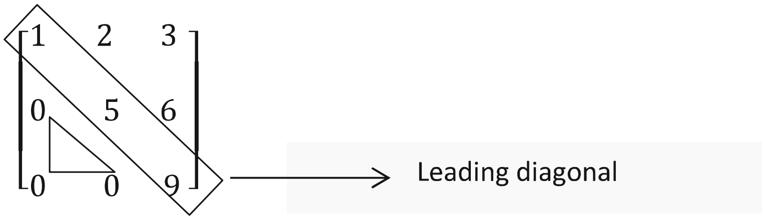
A is a square matrix of order 3 x 3.

5. **Triangular Matrix:**

(a) Upper Triangular Matrix:

In a square matrix if all the elements below the leading diagonal are zero then it is called an upper triangular matrix.

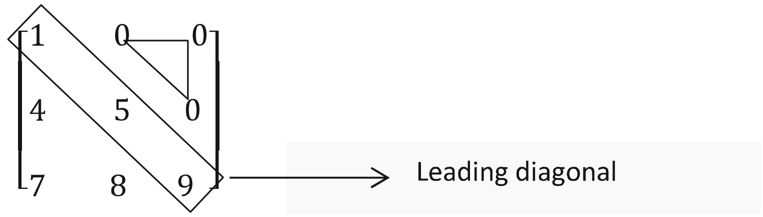
Eg. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$



(b) Lower Triangular Matrix:

In a square matrix if all the elements above the leading diagonal are zero, then it is called a Lower Triangular Matrix.

Eg. $B = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}$



6. **Transpose of Matrix**

Let A be any matrix. The transpose of matrix A is obtained by interchanging either rows into columns or columns into rows of A. It is denoted by A^T or A'.

Eg. If $A = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 6 \\ 2 & -1 & 9 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 5 & -1 \\ 3 & 6 & 9 \end{bmatrix}$

Note:

(i) If a matrix A is of order m x n then the order of A^T is n x m.

(ii) $(A^T)^T = A$.

7. Symmetric Matrix:

The square matrix A is called a symmetric matrix if $A = A^T$.

For Example:

$$\text{If } A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \quad \text{then } A^T = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$\therefore$ $A = A^T$

$\therefore$ A is symmetric.

8. Skew Symmetric Matrix:

The square matrix A is called a skew symmetric matrix if $A = -A^T$.

For Example:

$$\text{If } A = \begin{bmatrix} a & h & g \\ -h & b & f \\ -g & -f & c \end{bmatrix} \text{ then } -A^T = \begin{bmatrix} a & h & g \\ -h & b & f \\ -g & -f & c \end{bmatrix}$$

Here $A = -A^T$

$\therefore$ A is skew symmetric.

9. Diagonal Matrix:

In a square matrix, if all the elements other than the elements of the leading Diagonal (or) main diagonal are zero then the matrix is called Diagonal matrix.

$$\text{Eg.} \quad A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \xrightarrow{\quad} \text{Leading diagonal}$$

10. Scalar Matrix:

A diagonal matrix in which all the elements are equal to a scalar is called a scalar matrix.

$$\text{Eg.} \quad A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

11. Unit Matrix:

A square matrix in which all the elements of the leading diagonal are 1 and other elements are zero, is called a Unit Matrix.

It is denoted by I.

Eg. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2.

Operations on Matrices:

i) Addition and subtraction of matrices

ii) Multiplication of matrix by a scalar

iii) Multiplication of two matrices

i) Addition and Subtraction of Matrices:

Two Matrices can be added (or) subtracted provided both the matrices are of same order. We can add (or) subtract the corresponding elements of two matrices of same order.

Example: 1

If $A = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 4 & 5 \\ 3 & 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 1 & 7 & 5 \end{bmatrix}$ then

Find A + B

Solution:

$$A + B = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 4 & 5 \\ 3 & 1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 1 & 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 2+3 & 7+1 \\ 0+2 & 4+4 & 5+0 \\ 3+1 & 1+7 & 6+5 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 8 \\ 2 & 8 & 5 \\ 4 & 8 & 11 \end{bmatrix}$$

Example: 2

If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & 7 \\ 1 & 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 & 4 \\ 1 & -1 & 5 \\ 0 & 2 & 4 \end{bmatrix}$ then

Find A - B

Solution:

$$A - B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & 7 \\ 1 & 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 3 & 4 \\ 1 & -1 & 5 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1-7 & 3-3 & 5-4 \\ 2-1 & 0-(-1) & 7-5 \\ 1-0 & 5-2 & 2-4 \end{bmatrix}$$

$$A - B \Rightarrow \begin{bmatrix} -6 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & -2 \end{bmatrix}$$

ii) Multiplication of a matrix by a scalar:

We can multiply the matrix by any non-zero scalar K. To multiply the matrix by a scalar K, multiply all the elements by the same scalar K.

i.e., If $A = [a_{ij}]_{m\times n}$ then $KA = [Ka_{ij}]_{m\times n}$

for all $i = 1, 2, \ldots.. m$

$j = 1, 2, \ldots.. n$

For Example:

$$\text{If } A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 1 & 0 \\ 7 & 2 & 8 \end{bmatrix} \;\&\; B = \begin{bmatrix} -3 & 1 & 0 \\ 2 & 7 & 1 \\ 4 & 3 & 5 \end{bmatrix}$$

Then find 2A and 7B

Solution:

$$\text{Given } A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 1 & 0 \\ 7 & 2 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 1 & 0 \\ 2 & 7 & 1 \\ 4 & 3 & 5 \end{bmatrix}$$

$$2A = 2 \begin{bmatrix} 4 & 3 & 2 \\ 5 & 1 & 0 \\ 7 & 2 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 \text{ x } 4 & 2 \text{ x } 3 & 2 \text{ x } 2 \\ 2 \text{ x } 5 & 2 \text{ x } 1 & 2 \text{ x } 0 \\ 2 \text{ x } 7 & 2 \text{ x } 2 & 2 \text{ x } 8 \end{bmatrix}$$

$$2A \Rightarrow \begin{bmatrix} 8 & 6 & 4 \\ 10 & 2 & 0 \\ 14 & 4 & 16 \end{bmatrix}$$

$$7B = 7 \begin{bmatrix} -3 & 1 & 0 \\ 2 & 7 & 1 \\ 4 & 3 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 7 \text{ x } -3 & 7 \text{ x } 1 & 7 \text{ x } 0 \\ 7 \text{ x } 2 & 7 \text{ x } 7 & 7 \text{ x } 1 \\ 7 \text{ x } 4 & 7 \text{ x } 3 & 7 \text{ x } 5 \end{bmatrix}$$

$$7B = \begin{bmatrix} -21 & 7 & 0 \\ 14 & 49 & 7 \\ 28 & 21 & 35 \end{bmatrix}$$

Example: 2

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 7 \\ 0 & 4 \end{bmatrix} \text{ then find } 4A - 2B$$

Solution:

$$\text{Given } A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 7 \\ 0 & 4 \end{bmatrix}$$

$$4A = 4\begin{bmatrix}1 & 2\\3 & 5\end{bmatrix}$$

$$4A \Rightarrow \begin{bmatrix}4 & 8\\12 & 20\end{bmatrix} \qquad \text{------- (1)}$$

$$2B = 2\begin{bmatrix}-5 & 7\\0 & 4\end{bmatrix}$$

$$2B = \begin{bmatrix}-10 & 14\\0 & 8\end{bmatrix} \qquad \text{------- (2)}$$

$$(1) - (2) \Rightarrow \quad 4A - 2B = \begin{bmatrix}4 & 8\\12 & 20\end{bmatrix} - \begin{bmatrix}-10 & 14\\0 & 8\end{bmatrix}$$

$$4A - 2B \Rightarrow \begin{bmatrix}14 & -6\\12 & 12\end{bmatrix}$$

iii) Multiplication of Matrices:

Let A and B be any two Matrices. Multiplication of two matrices possible only when the number of columns of A must be equal to the number of rows of B.

Let A, B, C be any three matrices of same order.

$$\text{i.e} \quad A = \begin{bmatrix}a_1 & b_1\\a_2 & a_2\end{bmatrix}, \qquad B = \begin{bmatrix}p_1 & q_1\\p_2 & q_2\end{bmatrix}, \qquad C = \begin{bmatrix}x_1 & y_1\\x_2 & y_2\end{bmatrix}$$

$$\text{then, i) } AB = \begin{bmatrix}a_1 & & b_1\\ & \longrightarrow & \\a_2 & & b_2\end{bmatrix}\begin{bmatrix}p_1 & \big| & q_1\\ & & \\p_2 & \downarrow & q_2\end{bmatrix}$$

$$AB \Rightarrow \begin{bmatrix}a_1p_1 + b_1p_2 & a_1q_1 + b_1q_2\\ & \\a_2p_1 + b_2p_2 & a_2q_1 + b_2q_2\end{bmatrix}$$

$$\text{ii) } BC = \begin{bmatrix}p_1 & & q_1\\ & \longrightarrow & \\p_2 & & q_2\end{bmatrix}\begin{bmatrix}x_1 & \big| & y_1\\ & & \\x_2 & \downarrow & y_2\end{bmatrix}$$

$$BC \Rightarrow \begin{bmatrix}p_1x_1 + q_1x_2 & p_1y_1 + q_1y_2\\ & \\p_2x_1 + q_2x_2 & p_2y_1 + q_2y_2\end{bmatrix}$$

Example: 1

$$\text{If } A = \begin{bmatrix}1 & -1\\1 & 2\end{bmatrix}, \qquad B = \begin{bmatrix}3 & 4\\5 & 0\end{bmatrix} \text{ then find AB}$$

Solution:

$$AB = \begin{bmatrix}1 & & -1\\ & \longrightarrow & \\1 & & 2\end{bmatrix}\begin{bmatrix}3 & \big| & 4\\ & & \\5 & \downarrow & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}(1 \text{ x } 3) + (-1 \text{ x } 5) & (1 \text{ x } 4) + (-1 \text{ x } 0)\\ & \\(1 \text{ x } 3) + (2 \text{ x } 5) & (1 \text{ x } 4) + (2 \text{ x } 0)\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}3 - 5 & 4 + 0\\ & \\3 + 10 & 4 + 0\end{bmatrix}$$

$$AB \Rightarrow \begin{bmatrix} -2 & 4 \\ 13 & 4 \end{bmatrix}$$

Example: 2

If $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ verify AB = BA

Solution:

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1 & 0+3 \\ 2+3 & 0+9 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 3 \\ 5 & 9 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4+0 & 2+0 \\ 2+3 & 1+9 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 2 \\ 5 & 10 \end{bmatrix}$$

$\therefore$ AB $\neq$ BA

Example: 3

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 7 & 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 3 & -1 \\ 2 & -4 & 4 \\ 5 & 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} -4 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 1 & 1 \end{bmatrix}$

then find A (B + C)

Solution:

$$B + C = \begin{bmatrix} 8 & 3 & -1 \\ 2 & -4 & 4 \\ 5 & 3 & 1 \end{bmatrix} + \begin{bmatrix} -4 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 & -1 \\ 2 & -1 & 8 \\ 10 & 4 & 2 \end{bmatrix}$$

$$A(B + C) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 7 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 5 & -1 \\ 2 & -1 & 8 \\ 10 & 4 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4+4+30 & 5-2+12 & -1+16+6 \\ 0+10+40 & 0-5+16 & 0+40+8 \\ 28+4+40 & 35-2+16 & -7+16+8 \end{bmatrix}$$

$$A(B + C) = \begin{bmatrix} 38 & 15 & 21 \\ 50 & 11 & 48 \\ 72 & 49 & 17 \end{bmatrix}$$

Exercise

(1) If $A = \begin{bmatrix} 1 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 7 & -2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$ & $C = \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$

then verify (AB) C = A(BC)

(2) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}$

verify that A (B + C) = AB + BC

Properties of Matrices:

i) Commutative Property in Matrix Addition:

For any matrices A & B of same order,

a) Matrix Addition is commutative:

A + B = B + A ,

b) Matrix Multiplication is not commutative in general.

$\therefore$ AB $\neq$ BA

ii) a) **Associative Property in Matrix Addition:**

(A + B) + C = A + (B + C), for any matrices A, B & C of same order.

b) **Associative Property in Matrix Multiplication:**

A(BC) = (AB)C , for any matrices A, B & C such that A(BC) and (AB)C are of same order.

iii) a) Identity Property in Matrix Addition:

For any matrix A, there exist a matrix O of same order such that A + O = O + A = A

b) Identity Property in Matrix Multiplication:

For any matrix A, there exist an Identity matrix I such that AI = IA = A

Here, I is a Unit Matrix.

iv) Inverse Property:

For any matrix A, – A is the additive inverse of A such that A + (-A) = O = (-A) + A

v) Distributive Property:

a) Matrix multiplication is left distributive over addition

A (B + C) = AB + AC

b) Matrix multiplication is right distributive over addition

(A + B) C = AC + BC

c) Scalar multiplication is distributive over addition

$(a + b) A = aA + bA$

$a (A + B) = aA + aB$

Here, a, b are any scalar

A, B are any matrices of same order.

Worked Examples

1) If $A = \begin{bmatrix} 3 & 4 \\ 8 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix}$

Then verify that i) A + B = B + A

ii) A + (- A = (-A) + A = O

Solution:

i) $A = \begin{bmatrix} 3 & 4 \\ 8 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix}$

$$A + B = \begin{bmatrix} 3 & 4 \\ 8 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & 7 \\ 9 & -3 \end{bmatrix} \qquad \text{----- (1)}$$

$$B + A = \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 8 & -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & 7 \\ 9 & -3 \end{bmatrix} \qquad \text{----- (2)}$$

$\therefore$ A + B = B + A

ii) $A = \begin{bmatrix} 3 & 4 \\ 8 & -3 \end{bmatrix}$, $-A = \begin{bmatrix} -3 & -4 \\ -8 & 3 \end{bmatrix}$

$$A + (-A) = \begin{bmatrix} 3 & 4 \\ 8 & -3 \end{bmatrix} + \begin{bmatrix} -3 & -4 \\ -8 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \qquad \text{----- (1)}$$

$$-A + A = \begin{bmatrix} -3 & -4 \\ -8 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 8 & -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \qquad \text{----- (2)}$$

$\therefore$ (1) = (2)

A + (-A) = (-A) + A = 0

2) If $A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 2 & 0 \\ 1 & 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -1 & 2 \\ 4 & 1 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 8 & 3 & 1 \\ 0 & 5 & 4 \\ 1 & 2 & 3 \end{bmatrix}$

Then verify that A + (B + C) = (A + B) + C

Solution:

Given,

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 2 & 0 \\ 1 & 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -1 & 2 \\ 4 & 1 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 8 & 3 & 1 \\ 0 & 5 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

L.H.S.

$$B + C = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -1 & 2 \\ 4 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 8 & 3 & 1 \\ 0 & 5 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

$$B + C = \begin{bmatrix} 10 & 6 & 2 \\ -1 & 4 & 6 \\ 5 & 3 & 8 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 2 & 0 \\ 1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 10 & 6 & 2 \\ -1 & 4 & 6 \\ 5 & 3 & 8 \end{bmatrix}$$

$$A + (B + C) \Rightarrow \begin{bmatrix} 14 & 9 & 3 \\ 1 & 6 & 6 \\ 6 & 6 & 13 \end{bmatrix} \qquad \text{----- (1)}$$

R.H.S.

$$A + B = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 2 & 0 \\ 1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ -1 & -1 & 2 \\ 4 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 & 2 \\ 1 & 1 & 2 \\ 5 & 4 & 10 \end{bmatrix}$$

$$(A + B) + C = \begin{bmatrix} 6 & 6 & 2 \\ 1 & 1 & 2 \\ 5 & 4 & 10 \end{bmatrix} + \begin{bmatrix} 8 & 3 & 1 \\ 0 & 5 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

$$(A + B) + C) = \begin{bmatrix} 14 & 9 & 3 \\ 1 & 6 & 6 \\ 6 & 6 & 13 \end{bmatrix} \qquad \text{----- (2)}$$

(1) = (2)

$\therefore A + (B + C) = (A + B) + C$

Hence proved.

Exercise Problems

1) Verify the property A (B + C) = AB + AC for the following matrices A, B and C.

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$

2) Check the Associative property of matrix multiplication to the following matrices A, B, C.

$$A = \begin{bmatrix} 5 & 0 \\ 4 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ 5 & 3 \end{bmatrix}$$

3) If $A = [2 \;\; 3 \;\; -1]$ $B = \begin{bmatrix} 3 & 4 \\ 1 & 0 \\ 5 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 7 \\ 0 & -1 \end{bmatrix}$

Show that (AB) C = A (BC)

4) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. $B = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}$. $C = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

Show that (A – B) C = AC – BC

5) Show that the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$ Satisfy commutative property

AB = BA.

Reducing a Matrix into Triangular and Row Echelon Form:

Using the row elementary operations, We can transform a given non zero matrix to a simplified form called a Row-echelon Form.

In a Row – echelon form, we may have rows all of whose entries are zero, such rows are called zero rows.

Simply said,

If a non zero matrix is in row-echelon form, then all the entries below the leading diagonal [ie., a_{11}, a_{22}, a_{33} ....] are zeros.

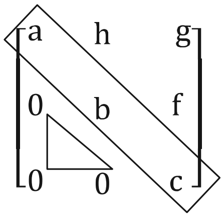
As similar way to propose for the triangular form.

There is two type of triangular form.

i.e. Upper Triangular Form:

If a matrix is said to be a Upper Triangular Form which all the elements below the leading diagonal are zero.

Eg. A =



$$A = \begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$$

i.e. Lower Triangular Form:

If a matrix is said to be a Lower Triangular Form which all the elements above the leading diagonal are zero.

Eg. $$B = \begin{bmatrix} a & 0 & 0 \\ e & b & 0 \\ f & h & c \end{bmatrix}$$

Worked Problem

1) Reduce the Matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ to a row-echelon form.

Solution:

Given, $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

$R_2 \to R_2 + 2R$, $R_3 \to R_3 + R_1$

$$\Rightarrow \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

This is the required Row – echelon form.

2) Reduce the matrix $\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$ into Row-echelon form.

Solution:

Given, $\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 4 & 2 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 4R_1$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 2 & 8 & 20 \end{bmatrix}$$

$R_3 \to 3R_3 - 2R_2$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 22 & 48 \end{bmatrix}$$

This is the required row-echelon form.

3) Reduce the matrix $\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$ into a Triangular Form.

Solution:

Given, $\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & -2 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - 3R_1$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 7 & 1 \\ 0 & 5 & -5 \end{bmatrix}$$

$R_3 \rightarrow 7R_3 - 5R_2$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 7 & 1 \\ 0 & 0 & -40 \end{bmatrix}$$

This is the required Triangular form.

4) Reduce the matrix $\begin{bmatrix} 4 & 3 & 6 & 25 \\ 1 & 5 & 7 & 13 \\ 2 & 9 & 1 & 1 \end{bmatrix}$ into a Triangular Form.

Solution:

Given, $\begin{bmatrix} 4 & 3 & 6 & 25 \\ 1 & 5 & 7 & 13 \\ 2 & 9 & 1 & 1 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 7 & 13 \\ 4 & 3 & 6 & 25 \\ 2 & 9 & 1 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 4R_1, \quad R_3 \rightarrow R_3 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & -17 & -22 & -27 \\ 0 & -1 & -13 & -25 \end{bmatrix}$$

$R_2 \rightarrow R_2 \div (-1), \quad R_3 \rightarrow R_3 \div (-1)$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & +17 & +22 & +27 \\ 0 & 1 & 13 & 25 \end{bmatrix}$$

$R_3 \rightarrow 17R_3 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & 17 & 22 & 27 \\ 0 & 0 & 199 & 398 \end{bmatrix}$$

This is the required Triangular Form.

Exercise

(1) Reduce the following Matrix into Row-echelon Form.

a) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(2) Reduce the following Matrix into Triangular Form.

c) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

d) $\begin{bmatrix} 3 & 3 & -1 & 11 \\ 2 & -1 & 2 & 9 \\ 4 & 3 & 2 & 25 \end{bmatrix}$

Transpose of a matrix and its properties:

The transpose of a matrix is obtained by interchanging their rows and columns of the matrix A and it is denoted by A^T.

More precisely, if $A = [a_{ij}]_{m\times n}$

Then $A^T = [b_{ij}]_{n\times m}$, where $bij = aji$

For instance,

$$A = \begin{bmatrix} 1 & \sqrt{2} & 4 \\ -8 & 0 & 0.2 \end{bmatrix} \text{ implies } A^T = \begin{bmatrix} 1 & -8 \\ \sqrt{2} & 0 \\ 4 & 0.2 \end{bmatrix},$$ Here (i, j)th entry of A^T is aji.

Results on Transpose of a Matrix:

For any two matrices A and B of suitable orders

(i) $(A^T)^T = A$

(ii) $(KA)^T = KA^T$ (Where K is any scalar)

(iii) $(A + B)^T = A^T + B^T$

(iv) $(AB)^T = B^T A^T$ [Reversal law on Transpose]

Examples

1. If $A = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

Verify i) $(AB)^T = B^T A^T$ ii) $(A+B)^T = A^T + B^T$

iii) $(A-B)^T = A^T - B^T$ iv) $(3A)^T = 3A^T$

Solution:

i) $$AB = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 2 & 22 \\ -2 & 9 & 9 \\ 7 & 1 & 14 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 16 & -2 & 7 \\ 2 & 9 & 1 \\ 22 & 9 & 14 \end{bmatrix} \qquad \text{-----}\ (1)$$

$$B^T = \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -2 & 7 \\ 2 & 9 & 1 \\ 22 & 9 & 14 \end{bmatrix} \qquad \text{-----}\ (2)$$

From (1) and (2),

$(AB)^T = B^TA^T$

ii) $$A + B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 1 \\ 3 & 0 & 9 \\ -1 & 5 & 3 \end{bmatrix}$$

$$(A + B)^T = \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \qquad \text{-----}\ (3)$$

$$A^T + B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \qquad \text{-----}\ (4)$$

From (3) and (4),

$(A+B)^T = A^T + B^T$

iii) $$A - B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 & 3 \\ -3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$(A\text{-}B)^T = \begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \qquad \text{----- (5)}$$

$$A^T - B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \qquad \text{----- (6)}$$

From (5) and (6), $(A - B)^T = A^T - B^T$

iv) $$3A = \begin{bmatrix} 12 & 18 & 6 \\ 0 & 3 & 15 \\ 0 & 9 & 6 \end{bmatrix}$$

$$(3A)^T = \begin{bmatrix} 12 & 0 & 0 \\ 18 & 3 & 9 \\ 6 & 15 & 6 \end{bmatrix} = 3\begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$$

$$= 3(A^T)$$

2. If $A = \begin{bmatrix} 2 & -3 & 8 \\ 21 & 6 & -6 \\ 4 & -33 & 19 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -29 & -8 \\ 2 & 0 & 3 \\ 17 & 15 & 4 \end{bmatrix}$

Verify (i) $(A+B)^T = A^T+B^T$ (ii) $(AB)^T = B^T A^T$

i) $$A + B = \begin{bmatrix} 2 & -3 & 8 \\ 21 & 6 & -6 \\ 4 & -33 & 19 \end{bmatrix} + \begin{bmatrix} 1 & -29 & -8 \\ 2 & 0 & 3 \\ 17 & 15 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -32 & 0 \\ 23 & 6 & -3 \\ 21 & -18 & 23 \end{bmatrix}$$

$$(A + B)^T = \begin{bmatrix} 3 & 23 & 21 \\ -32 & 6 & -18 \\ 0 & -3 & 23 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 21 & 4 \\ -3 & 6 & -33 \\ 8 & -6 & 19 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & 2 & 17 \\ -29 & 0 & 15 \\ -8 & 3 & 4 \end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix} 2 & 21 & 4 \\ -3 & 6 & -33 \\ 8 & -6 & 19 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 17 \\ -29 & 0 & 15 \\ -8 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 23 & 21 \\ -32 & 6 & -18 \\ 0 & -3 & 23 \end{bmatrix}$$

So we can observe that $(A + B)^T = A^T + B^T$.

ii)

$$AB = \begin{bmatrix} 2 & -3 & 8 \\ 21 & 6 & -6 \\ 4 & -33 & 19 \end{bmatrix} + \begin{bmatrix} 1 & -29 & -8 \\ 2 & 0 & 3 \\ 17 & 15 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 132 & 62 & 7 \\ -69 & -699 & -174 \\ 261 & 169 & -55 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 132 & -69 & 261 \\ 62 & -699 & 169 \\ 7 & -174 & -55 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 2 & 17 \\ -29 & 0 & 15 \\ -8 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 21 & 4 \\ -3 & 6 & -33 \\ 8 & -6 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} 132 & -69 & 261 \\ 62 & -699 & 169 \\ 7 & -174 & -55 \end{bmatrix}$$

$\therefore (AB)^T = B^T A^T$.

3. If $A = \begin{bmatrix} 9 & 8 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}$

Show that i) $(AB)^T = B^T A^T$ ii) $(AB)^T \neq A^T B^T$

Solution:

$$AB = \begin{bmatrix} 9 & 8 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 44 & 18 \\ 5 & 4 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 44 & 5 \\ 18 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 9 & 2 \\ 8 & -3 \end{bmatrix}, \qquad B^T = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix}$$

i) $B^T A^T = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 9 & 2 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 44 & 5 \\ 18 & 4 \end{bmatrix}$

$B^T A^T = (AB)^T$

ii) $A^T B^T = \begin{bmatrix} 9 & 2 \\ 8 & -3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 40 & 9 \\ 26 & 8 \end{bmatrix}$

$A^T B^T \neq (AB)^T$

We can clearly observe from here that $(AB)^T \neq A^T B^T$

Exercise

1. If $A = \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 7 \\ 9 & -1 \\ 1 & -2 \end{bmatrix}$

Verify the following

i) $(A + B)^T = A^T + B^T$ ii) $(A - B)^T = A^T - B^T$

iii) $(B^T)^T = B$

2. If $P = \begin{bmatrix} 2 & 8 & 9 \\ 11 & -15 & -13 \end{bmatrix}$ and K is a constant, then verify $(KP)^T = KP^T$
3. If A is a 3 x 4 matrix and B is a matrix such that both A^TB and BA^T are defined, what is the order of the matrix B.

Determinants:

Introduction of Determinants

The method of solving simultaneous linear equations was instrumental to the origin of the topic determinants. The theory of determinants began with Leibnitz who solved, the simultaneous linear equation.

Definition of a Determinant:

The determinant is a scalar value that can be computed from the elements of a square matrix A.

It is denoted by det (A)

Also $\Delta A = \det(A) = |A|$

First Order Determinant:

Let A = [a] be the matrix of order 1 Then the determinant of A is defined as “a”.

Second Order Determinant:

If a Determinant consists of two rows and two columns then it is called a second order determinant.

Ex: $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Third Order Determinant:

If a Determinant consist of three rows and three columns then it is called a third order determinant.

Ex: $|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -1 & 0 \\ 1 & -2 & 3 \end{vmatrix} = 1(-3 + 0) + 1(6 - 0) + 2(-4 + 1)$

$|A| = -3$

Properties of Determinants:

Property: 1

The value of the determinant is unaltered by changing rows into columns and vice versa. i.e. $|A| = |A^T|$

Proof:

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Rightarrow a_1 (b_2 c_3 - c_2 b_3) - b_1 (a_2 c_3 - c_2 a_3) + c_1 (a_2 b_3 - a_3 b_2)$$

$$|A^T| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 (b_2 c_3 - c_2 b_3) - a_2 (b_1 c_3 - c_1 b_3) + a_3 (b_1 c_2 - c_1 b_2)$$

$$= a_1 (b_2 c_3 - c_2 b_3) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$$

$$|A| = |A^T|$$

Proved.

Property: 2

If any two rows / columns of a determinant are interchanged, then the value of the determinant changes in sign but its absolute value remains unaltered.

$$\text{Let} \quad |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$|A| = a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$$

$R_2 \leftrightarrow R_3$

$$|A_1| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= a_1 (b_3 c_2 - b_2 c_3) - b_1 (a_3 c_2 - a_2 c_3) + c_1 (a_3 b_2 - a_2 b_3)$$

$$= -a_1 (b_2 c_3 - b_3 c_2) + b_1 (a_2 c_3 - a_3 c_2) - c_1 (a_2 b_3 - a_3 b_2)$$

$$= -[a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)]$$

$$|A_1| = -|A|$$

Hence proved.

Note:

If there are n interchanges of rows (columns) of a matrix A then the resulting determinant is $(-1)^n |A|$.

Property: 3

If any two rows (or) two columns of a matrix are identical, then the value of the determinant is zero.

Proof:

Let $|A| = \begin{vmatrix} 2 & 5 & 1 \\ 2 & 5 & 1 \\ 0 & 2 & 4 \end{vmatrix}$

Here R_1, R_2 are identical.

$$|A| = 2(20-2) - 5(8-0) + 1(4-0)$$

$$= 2(18) - 5(8) + 4$$

$$= 36 - 40 + 4$$

$$|A| = 0$$

Property: 4

If each element of a row (or column) is multiplied by any scalar K, then the value of the determinant is also multiplied by the same scalar K.

i.e. If $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

then $|A_1| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Here R_1 is multiplied K

$$\therefore |A_1| = k|A|$$

Consider

$$|A| = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 2 & 4 \\ 3 & 9 & 4 \end{vmatrix}$$

$$\Rightarrow \quad 1(8-36) - 0(-4-12) + 1(-9-6)$$

$$= \quad -28 - 15 = -43$$

Multiply the Row 1 by 2

$$|A_1| = \begin{vmatrix} 2 & 0 & 2 \\ -1 & 2 & 4 \\ 3 & 9 & 4 \end{vmatrix}$$

$$\Rightarrow 2(8-36) - 0(-4-12) + 2(-9-6)$$

$$\Rightarrow 2(-28) - 0 + 2(-15)$$

$$= -56 - 30$$

$$= -86$$

$$= 2(-43)$$

$$|A_1| = 2|A|$$

Property: 5

If each element of a row (or column) of a determinant is expressed as sum of two or more terms then the whole determinant can be expressed as the sum of two (or) more determinants of the same order.

i.e. $$\begin{vmatrix} a_1 & b_1 + m_1 & c_1 \\ a_3 & b_2 + m_2 & c_3 \\ a_2 & b_3 + m_3 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & m_1 & c_1 \\ a_2 & m_2 & c_2 \\ a_3 & m_3 & c_3 \end{vmatrix}$$

Property: 6

A determinant is unaltered when to each element of any row (or column) are added those of several other rows or columns multiplied respectively by constant factors.

i.e. if $$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$|A_1| = \begin{vmatrix} a_1 + pa_2 + qa_3 & b_1 + pb_2 + qb_3 & c_1 + pc_2 + qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then $|A| = |A_1|$

Proof:

$$|A_1| = \begin{vmatrix} a_1 + pa_2 + qa_3 & b_1 + pb_2 + qb_3 & c_1 + pc_2 + qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

By property (5),

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pa_2 & pb_2 & pc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} qa_3 & qb_3 & qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

By property (4)

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + p \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + q \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

By property (3)

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + p\,(0) + q\,(0)$$

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \therefore \; |A_1| \Rightarrow |A|$$

Problems using Properties of Determinants:

1. Evaluate: $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

Effect $C_2 = C_2 + C_3$

$$= \begin{vmatrix} 1 & a+b+c & b+c \\ 1 & b+c+a & c+a \\ 1 & c+a+b & a+b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}$$

$$= (a+b+c)\, 0$$

$$= 0 \quad \therefore [C_1 \equiv C_2)$$

2. Prove that $\begin{vmatrix} 2a+b & a & b \\ 2b+c & b & c \\ 2c+a & c & a \end{vmatrix} = 0$

Solution:

$$\text{LHS} = \begin{vmatrix} 2a+b & a & b \\ 2b+c & b & c \\ 2c+a & c & a \end{vmatrix}$$

$$= \begin{vmatrix} 2a & a & b \\ 2b & b & c \\ 2c & c & a \end{vmatrix} + \begin{vmatrix} b & a & b \\ c & b & c \\ a & c & a \end{vmatrix}$$

$$= 2\begin{vmatrix} a & a & b \\ b & b & c \\ c & c & a \end{vmatrix} + 0 \qquad [\because C_1 \equiv C_3]$$

$$= 2\,(0) \quad [\because C_1 \equiv C_2]$$

$$= 0$$

$$= \text{RHS}$$

3. Prove that $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = (x-y)(y-z)(z-x)$

Solution:

$$\text{LHS} = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

$R_1 \to R_1 - R_2$

$R_2 \to R_2 - R_3$

$$= \begin{vmatrix} 0 & x-y & yz-zx \\ 0 & y-z & zx-xy \\ 1 & z & xy \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x-y & -z(x-y) \\ 0 & y-z & -x(y-z) \\ 1 & z & xy \end{vmatrix}$$

$$= 0 + 0 + 1 \quad \begin{vmatrix} x-y & -z(x-y) \\ y-z & -x(y-z) \end{vmatrix}$$

(Expand along the first column)

$$= (x - y)\,(y - z) \quad \begin{vmatrix} 1 & -z \\ 1 & -x \end{vmatrix}$$

$$= (x - y)\,(y - z)\,(z - x)$$

$= \text{RHS}$

Product of Determinants:

While multiplying two matrices "row-by column" rule alone can be followed. The process of interchanging the rows and columns will not affect the value of the determinant i.e. we can also apply the following procedures for multiplication of two determinants.

(i) Row by row multiplication rule

(ii) Row by column multiplication rule

(iii) Column by column multiplication rule

(iv) Column by row multiplication rule

Note:

If A and B are square matrices of same order n, then $|AB| = |A|\,|B|$.

In Matrices, $AB \neq BA$ in general, we can $|AB| = |BA|$

Worked Examples

(1) If $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$, $|B| = \begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix}$, then find the product of Determinant?

Solution:

By Row – Column Multiple,

$$|A|\,|B| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+6 & 0-4 \\ 3+12 & 0-8 \end{vmatrix}$$

$$|A|\,|B| = \begin{vmatrix} 7 & -4 \\ 15 & -8 \end{vmatrix}$$

We know $|A|\,|B| = |AB|$

$$\text{So } |AB| = \begin{vmatrix} 7 & -4 \\ 15 & 8 \end{vmatrix}$$

(2) If A_i, B_i, C_i are the co-factors of a_i, b_i, c_i respectively, i = 1 to 3 in

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ show that } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2$$

Solution:

Consider the product

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

By the Row – Row Multiplication

$$\Rightarrow \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 + & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

$$\Rightarrow |A| [|A|^2 - 0]$$

$$\Rightarrow |A|^3$$

i.e. $$|A| \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^3$$

$$\Rightarrow \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2$$

Hence proved.

(3) Verify that, $|AB| = |A||B|$ if $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Solution:

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \cos\theta\sin\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 1 - 0$$

$$|AB| = 1$$

$$|A| = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

$$|A| = \cos^2\theta + \sin^2\theta$$

$$|A| = 1$$

$$|B| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

$$= \cos^2\theta + \sin^2\theta$$

$$|B| = 1$$

$$\therefore |A||B| = 1 . 1 = 1$$

$$\because |A||B| = 1 . 1 = 1$$

Hence proved.

Exercise

1) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then find the product of determinants by Row – Row Multiplication.

2) If $A = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$. $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ then find | AB |

3) Show that $\begin{vmatrix} o & c & b \\ c & o & a \\ b & a & o \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$

Chapter 1.2 APPLICATIONS OF MATRICES AND DETERMINANTS

Minor of an element:

Minor of an element is the determinant obtained by deleting the row and column in which that element occurs.

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Minor of $a_1 \Rightarrow \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

Minor of $a_1 \Rightarrow b_2 c_3 - c_2 b_3$

Example: 1

Find the minor of 2 to the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \\ 7 & 8 & -2 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \\ 7 & 8 & -2 \end{vmatrix}$

$\because$ Minor of 2 $= \begin{vmatrix} 0 & -1 \\ 8 & -2 \end{vmatrix}$

$= 0 + 8$

Minor of 2 = 8

Co-factor of an Element:

Co-factor of an element is defined as the signed minor.

$\because$ Co-factor of aij $= (-1)^{i+j}$ minor of a_{ij}

Here, a_{ij} is an element which is i^th^ row and j^th^ column.

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Co-factor of $a_1 = (-1)^{1+1}$ Minor of a_1

$= (-1)^2 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

$= (+1)\,[b_2 c_3 - b_3 c_2]$

Co-factor of $a_1 = b_2 c_3 - b_3 c_2$

Adjoint of Matrix:

The Adjoint of a square matrix A is the transpose of the matrix which is formed by replacing each element with the corresponding cofactor.

Method to find adjoint of Matrix of order 3 (order 2)

If A is square Matrix of order 3 (order 2)

i) Find the co-factor of all the elements of A.

ii) Form the matrix by replacing all the elements of A by the corresponding co-factor of A.

iii) Then take the Transpose of that matrix, then we get adj. A.

Example: 1 Find the Adjoint of the matrix $\begin{bmatrix} 1 & 3 \\ 6 & 5 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 3 \\ 6 & 5 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 3 \\ 6 & 5 \end{vmatrix} = 5 - 18 = -13$

Co-factor of Matrix A:

Co-factor of 1 = $(-1)^{1+1}5 = 5$

Co-factor of 3 = $(-1)^{1+2}6 = -6$

Co-factor of 6 = $(-1)^{2+1}3 = -3$

Co-factor of 5 = $(-1)^{2+2}1 = 1$

$\therefore$ Co-factor matrix $= \begin{bmatrix} 5 & -6 \\ -3 & 1 \end{bmatrix}$

Adj A $= \begin{bmatrix} 5 & -3 \\ -6 & 1 \end{bmatrix}$

Example: 2

Find the adjoint of the matrix $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$

Solution:

$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{vmatrix}$

$\Rightarrow 2(4 - 3) - 3(2 + 3) + 4(1 + 2)$

$\Rightarrow 2 - 3(5) + 4(3)$

$\Rightarrow 2 - 15 + 12$

$|A| = -1 \neq 0$

Co-factor 2 = $(-1)^{1+1}(4 - 3) = 1$

Co-factor 3 = $(-1)^{1+2}(2 + 3) = -5$

Co-factor 4 = $(-1)^{1+3}(1 + 2) = 3$

Co-factor 1 = $(-1)^{2+1}(6 - 4) = -2$

Co-factor 2 = $(-1)^{2+2}(4 + 4) = 8$

Co-factor 3 $\Rightarrow (-1)^{2+3}(2+3) = -5$

Co-factor -1 = $(-1)^{3+1}(9-8) = 1$

Co-factor of 1 = $(-1)^{3+2}(6-4) = -2$

Co-factor of 2 = $(-1)^{3+3}(4-3) = 1$

$$\text{Co-factor matrix} = \begin{bmatrix} 1 & -5 & 3 \\ -2 & 8 & -5 \\ 1 & -2 & 1 \end{bmatrix}$$

Adj A = [co-factor of A]^T

$$\text{Adj. A} = \begin{bmatrix} 1 & -2 & 1 \\ -5 & 8 & -2 \\ 3 & -5 & 1 \end{bmatrix}$$

Singular matrix and Non-singular matrix

A square matrix A is said to be singular matrix if $|A| = 0$.

A square matrix A is said to be non-singular matrix if $|A| \neq 0$.

Example: 1

Show that the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ is a non-singular matrix.

Solution:

$$\text{Let } A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$$

$$= 10 - 12$$

$$|A| = -2 \neq 0 \qquad \therefore \text{ A is non singular matrix .}$$

Example: 2

Prove that the matrix $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$ is a singular matrix.

Solution:

$$\text{Let } B = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix}$$

$$= 4 - 4$$

$$|B| = 0 \qquad \therefore \text{ B is a singular matrix.}$$

Inverse of a Matrix:

Let A be a non-singular matrix. If there exist a square matrix B, such that,

AB = BA = I then B is called the inverse of Matrix A.

Where I is the unit Matrix of same order

Also it is denoted by A^{-1}.

Inverse of square matrix A is defined as $A^{-1} = \frac{1}{|A|}$ adjA, $|A| \neq 0$

Note:

i) Inverse of a Matrix is unique

ii) $AA^{-1} = A^{-1}A = I$

iii) $(AB)^{-1} = B^{-1}A^{-1}$

iv) $(A^T)^{-1} = (A^{-1})^T$

Example: 1

If $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$, then find the inverse of A

Solution:

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}$$

$$= 4 + 3$$

$$|A| = 7$$

$$\text{adj } A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{7}\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

Example: 2

If $B = \begin{bmatrix} -1 & 2 \\ 1 & -5 \end{bmatrix}$ then find B^{-1}

Solution:

$$B = \begin{bmatrix} -1 & 2 \\ 1 & -5 \end{bmatrix}$$

$$|B| = \begin{vmatrix} -1 & 2 \\ 1 & -5 \end{vmatrix}$$

$$|B| = 5 - 2 = 3$$

$$\text{Adj } B = \begin{bmatrix} -5 & -2 \\ -1 & -1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \text{ adj } B$$

$$B^{-1} = \frac{1}{3}\begin{bmatrix} -5 & -2 \\ -1 & -1 \end{bmatrix}$$

3. Find the inverse of $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

Solution:

$$\text{Let } A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= 2(12 - 2) - 3(16 - 1) + 4(8 - 3)$$

$$= 20 - 45 + 20$$

$$|A| = -5 \neq 0$$

$\therefore$ Inverse of A exist.

Cofactors of Matrix A:

$$A_{11} = + \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 12 - 2 = 10$$

$$A_{12} = - \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = -(16 - 1) = -15$$

$$A_{13} = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 8 - 3 = 5$$

$$A_{21} = - \begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -(12 - 8) = -4$$

$$A_{22} = \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} = 8 - 4 = 4$$

$$A_{23} = - \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -(4 - 3) = -1$$

$$A_{31} = \begin{vmatrix} 3 & 4 \\ 3 & 1 \end{vmatrix} = 3 - 12 = -9$$

$$A_{32} = - \begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = -(2 - 16) = 14$$

$$A_{33} = \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6 - 12 = -6$$

$\therefore$ Adj (A) $= [Aij]^T$

$$= \begin{bmatrix} 10 & -15 & 5 \\ -4 & 4 & -1 \\ -9 & 14 & -6 \end{bmatrix}^T$$

$$= \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

$\therefore A^{-1} = \frac{1}{|A|}$ adjA

$$= \frac{1}{-5}\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

4. Find the inverse of $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 4 & 2 & 2 \end{bmatrix}$

Solution:

Let A $= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 4 & 2 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 4 & 2 & 2 \end{vmatrix}$

$= 2(0 - 4) - 1(2 - 8) + 1(2 - 0)$

$= 2(-4) - 1(-6) + 2$

$= -8+6+2$

$|A| = 0$

$\therefore$ A is a singular matrix $\Rightarrow$ Inverse of A does not exist.

Exercise

1) Find the Adjoint of the matrix $\begin{bmatrix} 5 & 2 \\ 10 & 4 \end{bmatrix}$

2) Find the Adjoint of the matrix $\begin{bmatrix} 2 & 5 & 7 \\ 7 & 1 & 6 \\ 5 & -4 & -1 \end{bmatrix}$

3) For any two Matrix A = $\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$ B = $\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$ prove that $(AB)^T = B^TA^T$

4) Find the inverse of $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & -3 \\ 6 & -2 & -1 \end{bmatrix}$

5) Find the inverse of $\begin{bmatrix} 5 & 0 \\ 2 & 3 \end{bmatrix}$

Rank of Matrix:

A positive integer ‘r’ is said to be the Rank of a non zero matrix if

(i) At least one minor of order r is non-zero.

(ii) All minors of higher order than r are zero. It is denote by ρ(A)

Note:

From the definition of rank of the matrix, it follows,

1. The rank of non-singular matrix of order n is n. If the matrix is singular, its rank is less than n.
2. The rank of a m x n matrix ‘A’ can at most be equal to the smaller of numbers m and n but it may be less.

   $\rho(A) \leq$ minimum of m and n.
3. If there is a non-zero minor of order ‘r’ then rank is ≥ r.
4. The rank of the null matrix is zero and rank of non zero matrix is ≥ 1.
5. The rank of In, the unit matrix of order n is equal to n.

   i.e. $\rho(In) = n$

   $\rho(I_2) = 2$

   $\rho(I_3) = 3$ etc.
6. $\rho(A) = \rho(A^T)$

Example

1) Find the rank of $\begin{bmatrix} 5 & 2 \\ 6 & 3 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 5 & 2 \\ 6 & 3 \end{bmatrix}$

Order of A = 2 x 2

$\therefore \rho(A) \leq 2.$

The highest order of minor of A = 2.

The minor is $\begin{vmatrix} 5 & 2 \\ 6 & 3 \end{vmatrix} = 15 - 12 = 3 \neq 0$

$\therefore$ Rank of A = ρ(A) = 2.

2) Find the rank of $\begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$

Order of A = 2 x 2

$\therefore \rho(A) \leq 2.$