MATHEMATICS

2nd SEM

All Branch

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depends upon

***Example – 1* : Show that $\lim_{x\to 1}(3x+2)=5$.**

***Solⁿ* :** Since, $(3x+2)-5=3x-3$,

$\therefore |(3x+2)-5|=3|x-1|$

Hence if $\in$ is given possible number, then

$|(3x+2)-5|<\in$ whenever $|x-1|<\in/3$

We can, therefore, choose $\delta=\dfrac{\in}{3}$, such that

$|(3x+2)-5|<\in$, whenever $0<|x-1|<\delta$.

Hence $\lim_{x\to 3}(3x+2)=5$

***Example – 2* : Show that $\lim_{x\to 3}\dfrac{x^2-9}{x-3}=6$**

***Solⁿ* :** The function is not defined at $x=3$. For $x\neq 3$,

$$\frac{x^2-9}{x-3}\equiv x+3 \text{ and hence } \left|\frac{x^2-9}{x-3}-6\right|=|(x+3)-6|=|x-3|.$$

Now $\left|\dfrac{x^2-9}{x-3}-6\right|<\in$, whenever $|x-3|<\in$.

Thus for given $\in>0$, we have found a positive number δ, which in this case, is equal to $\in$itself such that

$$0<|x-3|<\delta\Rightarrow\left|\frac{x^2-9}{x-3}-6\right|<\in.$$

Hence $\lim_{x\to 3}\dfrac{x^2-9}{x-3}=6$

valuation of Left Hand and Right Hand limits :

The statements $x\to a^-$ means that 'x' is tending to a from the left hand side i.e. x is a number less an a but very very close to 'a'. Therefore $x\to a^-$ is equivalent to $x=a-h$ where $h>0$ such that $h\to$. Simillarly $x\to a^+$ is equivalent to $x=a+h$. Where $h\to 0+$. Thus, we have the following rules for ding left hand and right hand limits at $x=a$.

ft Hand Limit :

To evaluate L.H.L of $f(x)$ at $x=a$ i.e.

$\lim_{x\to a^-}f(x)$ we proceed as follows :

Step 1 : Write $\lim\limits_{x \to a^-} f(x)$

Step 2 : Put x = a − h and replace $x \to a^-$ by

h → 0 to obtain $\lim\limits_{h \to 0} f(a-h)$

Step 3 : Simplify $\lim\limits_{h \to 0} f(a-h)$ by using the formula for the given function.

Step 4 : The value obtain in step III is the LHL of $f(x)$ at x = a.

Right Hand Limit :

To evaluate RHL of $f(x)$ at x = a i.e.

$\lim\limits_{x \to a^+} f(x)$ we proceed as followes :

Step 1 :Write $\lim\limits_{x \to a^+} f(x)$

Step 2 : Put x = a + h and replace $x \to a^+$ by

h → 0 to obtain $\lim\limits_{h \to 0} f(a+h)$

Step 3 : Simplify $\lim\limits_{h \to 0} f(a+h)$ by using the formula for the given function.

Step 4 : The value obtained in step III is the RHL of $f(x)$ at x = a.

Example – 1 : **Evaluate the left hand limit of the function.**

$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0 & , x = 4 \end{cases} \text{ at } x = 4.$$

Solⁿ : (LHL of $f(x)$ at $x = 4$)

$$= \lim_{x \to 4^-} f(x) = \lim_{h \to 0+} f(4-h) = \lim_{h \to 0+} \frac{|4-h-4|}{4-h-4} = \lim_{h \to 0+} \frac{|-h|}{-h} = \lim_{h \to 0+} \frac{-h}{h} = \lim_{h \to 0+} (-1) = -1$$

Example - 2 : **Evaluate the right hand limit of the function.**

$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0 & , x = 4 \end{cases} \text{ at } x = 4.$$

Solⁿ : (RHL of $f(x)$ at $x = 4$)

$$= \lim_{x \to 4^+} f(x) = \lim_{h \to 0+} f(4+h) = \lim_{h \to 0+} \frac{|4+h-4|}{4+h-4} = \lim_{h \to 0+} \frac{|h|}{h} = \lim_{h \to 0+} \frac{h}{h} = \lim_{h \to 0+} 1 = 1$$

limits of function defined by

So, $\underset{x\to1}{\text{Lim}}f(x)$

Example - 4 : If $f(x)=\begin{cases}\dfrac{x-|x|}{x}, & x\neq0\\ 2 & x=0\end{cases}$ at $x = 0$, So that $\underset{x\to0}{\text{Lim}}f(x)$ does not exist.

Solⁿ : We have,

(LHL of $f(x)$ at $x = 0$)

$$=\underset{x\to0^-}{\text{Lim}}f(x)=\underset{h\to0+}{\text{Lim}}f(0-h)=\underset{h\to0+}{\text{Lim}}\frac{-h-|-h|}{(-h)}=\underset{h\to0+}{\text{Lim}}\frac{-h-h}{-h}$$

$$=\underset{h\to0+}{\text{Lim}}\frac{-2h}{-h}=\underset{h\to0+}{\text{Lim}}\,2=2.$$

(RHL of $f(x)$ at $x = 0$)

$$=\underset{h\to0+}{\text{Lim}}f(x)=\underset{h\to0+}{\text{Lim}}f(0+h)=\underset{h\to0+}{\text{Lim}}\frac{h-|h|}{h}=\underset{h\to0+}{\text{Lim}}\frac{h-h}{h}$$

$$=\underset{h\to0}{\text{Lim}}\frac{0}{h}=\underset{h\to0}{\text{Lim}}\,0=0$$

Since, $\underset{x\to0^-}{\text{Lim}}f(x)\neq\underset{x\to0^+}{\text{Lim}}f(x)$. So, $\underset{x\to0}{\text{Lim}}f(x)$ does not exist.

Example - 5 : If $f(x)=\begin{cases}5x-4, & 0<x\leq1\\ 4x^3-3x, & 1<x<2\end{cases}$,

Show that $\underset{x\to1}{\text{Lim}}f(x)$ exists.

Solⁿ : We have :

(LHL of $f(x)$ at $x = 1$)

$$=\underset{x\to1^-}{\text{Lim}}f(x)=\underset{h\to0+}{\text{Lim}}f(1-h)=\underset{h\to0+}{\text{Lim}}\,5(1-h)-4=\underset{h\to0+}{\text{Lim}}\,1-5h=1$$

(RHL of $f(x)$ at $x = 1$)

$$=\underset{x\to1^+}{\text{Lim}}f(x)=\underset{h\to0+}{\text{Lim}}f(1+h)=\underset{h\to0+}{\text{Lim}}\,4(1+h)^3-3(1+h)$$

$$=4(1)^3-3(1)=1$$

Clearly, $\underset{x\to1-}{\lim}f(x)=\underset{x\to1+}{\lim}f(x)$. So, $\underset{x\to1}{\text{Lim}}f(x)$ exists and is equal to 1.

Hence $a = 1$ and $b = 2 - 3$

Expansion Formulaes :

Sometimes following expansion are very useful to evaluate limits.

(a) $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$

(b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^2}{3!} + \ldots\ldots\ldots$

(c) $a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \ldots\ldots$

(d) $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$

(e) $\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$

(f) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots.$

(g) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\ldots$

(h) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\ldots$

(i) $\sin^{-1}x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4}x^5 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}\frac{x^7}{7}$

(j) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots.$

(k) $\sec x = 1 + \frac{x^2}{2!} + 5\frac{x^4}{4!} + \ldots\ldots$

Theorem -2 : Prove that $\lim_{x\to 0}\left(\frac{a^x-1}{x}\right) = \log_e a, \ (a > 0)$

a point
2.1

Proof : $\lim_{x\to 0} \frac{a^x-1}{x}$

putting $a^x - 1 = y$, $a^x = y + 1$, $\log a^x = \log(y+1)$, $x \log a = \log(y+1)$

$x = \frac{\log(1+y)}{\log a}$ [When $x \to 0$ i.e. $y \to 0$]

$$= \lim_{y\to 0} \frac{y}{\frac{\log(1+y)}{\log a}} = \lim_{y\to 0} \frac{\log a}{\frac{\log(1+y)}{y}} = \frac{\log a}{\lim_{y\to 0}\frac{\log(1+y)}{y}} = \log_e a \left[\because \lim_{y\to 0}\frac{\log(1+y)}{y} = 1\right]$$

Another Method :

$$\lim_{x\to 0}\left(\frac{a^x-1}{x}\right) = \lim_{x\to 0}\left\{\frac{\{1+x(\log a)+\frac{x^2}{2!}(\log a)^2+.....\}-1}{x}\right\} \text{ [on expending } a^x\text{]}$$

$$= \lim_{x\to 0}\frac{x\left\{(\log a)+\frac{x}{2!}(\log a)^2+........\right\}}{x} = \lim_{x\to 0}\left\{\log a+\frac{x}{2!}(\log a)^2+........\right\}$$

$= \log a$ [Putting $x = 0$]

$$\therefore \lim_{x\to 0}\left(\frac{a^x-1}{x}\right) = \log_e a$$

Theorem - 3 : Prove that $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right) = 1$

Proof : $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right) = \lim_{x\to 0}\left\{\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+.....\right)-1}{x}\right\}$ [on expanding e^x]

$$= \lim_{x\to 0}\left\{\frac{\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+.....\right)}{x}\right\} = \lim_{x\to 0}\left\{\frac{x\left(1+\frac{x}{2!}+\frac{x^2}{3!}+.....\right)}{x}\right\}$$

$$= \lim_{x\to 0}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+........\right) \quad \therefore \lim_{x\to 0}\left(\frac{e^x-1}{x}\right) = 1$$

Example–1. Evaluate : (i) $\lim\limits_{x\to 0} \frac{\sin 2x}{x}$ (ii) $\lim\limits_{x\to 0} \frac{\sin 3x}{5x}$ (iii) $\lim\limits_{x\to 0} \left(\frac{\sin ax}{\sin bx}\right)$

Solⁿ : (i) $\lim\limits_{x\to 0} \frac{\sin 2x}{x} = \lim\limits_{x\to 0} \left(2 \times \frac{\sin 2x}{2x}\right) = 2 \cdot \lim\limits_{2x\to 0} \frac{\sin 2x}{2x}$

$[\because$ when $x \to 0$, then $2x \to 0]$

$= (2 \times 1) = 1$ $\left[\because \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right]$

(ii) $\lim\limits_{x\to 0} \frac{\sin 3x}{5x} = \lim\limits_{x\to 0} \left(\frac{3}{5} \cdot \frac{\sin 3x}{3x}\right) = \frac{3}{5} \cdot \lim\limits_{3x\to 0} \frac{\sin 3x}{3x}$

$[\because$ when $x \to 0$, then $3x \to 0]$

$= \left(\frac{3}{5} \times 1\right) = \frac{3}{5}$ $\left[\because \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right]$

(iii) $\lim\limits_{x\to 0} \left(\frac{\sin ax}{\sin bx}\right) = \lim\limits_{x\to 0} \frac{ax \cdot \left(\frac{\sin ax}{ax}\right)}{bx \cdot \left(\frac{\sin bx}{bx}\right)}$

$= \frac{a}{b} \cdot \lim\limits_{x\to 0} \frac{\left(\frac{\sin ax}{ax}\right)}{\left(\frac{\sin bx}{bx}\right)} = \frac{a}{b} \cdot \frac{\lim\limits_{ax\to 0} \frac{\sin ax}{ax}}{\lim\limits_{bx\to 0} \frac{\sin bx}{bx}} = \left(\frac{a}{b} \times \frac{1}{1}\right) = \frac{a}{b}$

$[\because x \to 0, \Rightarrow ax \to 0$ and $bx \to 0]$

$= \left(\frac{a}{b} \times \frac{1}{1}\right) = \frac{a}{b}$ $\left[\because \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right]$

Example–2 : **Evaluate :** $\lim\limits_{x\to 0} \frac{\sin 5x}{\tan 3x}$

Solⁿ : $\lim\limits_{x\to 0} \frac{\sin 5x}{\tan 3x} = \lim\limits_{x\to 0} \frac{5x \cdot \left(\frac{\sin 5x}{5x}\right)}{3x \cdot \left(\frac{\tan 3x}{3x}\right)} = \frac{5}{3} \cdot \frac{\lim\limits_{5x\to 0}\left(\frac{\sin 5x}{5x}\right)}{\lim\limits_{3x\to 0}\left(\frac{\tan 3x}{3x}\right)} = \left(\frac{5}{3} \times \frac{1}{1}\right) = \frac{5}{3}$

Example–3 : **Evaluate :** $\lim\limits_{x\to 0} \frac{\sin x^\circ}{x}$

Solⁿ : We know that $x^\circ = \left(\frac{\pi x}{180}\right)^c$

$\therefore \lim\limits_{x\to 0} \frac{\sin x^\circ}{x} = \lim\limits_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x} = \lim\limits_{x\to 0} \left\{\frac{\sin\left(\frac{\pi x}{180}\right)}{\left(\frac{\pi x}{180}\right)} \times \frac{\pi}{180}\right\}$

$= \frac{\pi}{180} \cdot \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta}$, (where $\theta = \frac{\pi x}{180}$) [clearly, $x \to 0 \Rightarrow \frac{\pi x}{180} \to 0 \Rightarrow \theta \to 0$]

$= \left(\frac{\pi}{180} \times 1\right) = \frac{\pi}{180}$.

$y \to 0 \qquad \partial y$

xample – 14 : Examine the existence of the following limits.

(a) $\lim\limits_{x\to 0} \frac{|x|}{x}$ **(b)** $\lim\limits_{x\to\infty} \frac{x}{|x|}$ **(c)** $\lim\limits_{x\to\infty} \frac{x^2 - x}{[x^2 - x]}$

ol^{n} : (a) $\lim\limits_{x\to 0-} \frac{|x|}{x} = \lim\limits_{x\to 0-} \frac{|-h|}{-h} = \lim\limits_{h\to 0} \frac{|-h|}{-h} = \lim\limits_{h\to 0} \frac{h}{-h} = -1$

$$\lim_{x\to 0+} \frac{|x|}{x} = \lim_{x\to 0+} \frac{|+h|}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

Hence $\lim\limits_{x\to 0} \frac{|x|}{x}$ does not exist.

(b) $\lim\limits_{x\to\infty} \frac{x}{[x]}$

We know $\forall x \in R \qquad [x] \le x < [x] + 1$

$$\Rightarrow \frac{[x]}{[x]} \le \frac{x}{[x]} < 1 + \frac{1}{[x]} = 1 \le \lim_{x\to\infty} \frac{x}{[x]} \le \lim_{x\to\infty} 1 + \frac{1}{[x]}$$

$= 1 \le \lim\limits_{x\to\infty} \frac{x}{[x]} \le 1$ [Hence $\lim\limits_{x\to\infty} \frac{x}{[x]} = 1$]

(c) $\lim\limits_{x\to\infty} \frac{x^2 - x}{[x^2 - x]}$ set $y = x^2 - x$ as $x \to \infty$, i.e. $y \to \infty$

$= \lim\limits_{y\to\infty} \frac{y}{[y]}$ exist's and equal to 1

Definition : A function $f(x)$ is said to be continuous at $x = a$ if :

(i) $\lim\limits_{x \to a} f(x)$ exists,

i.e. right hand and left hand limits exist and are equal.

(ii) $f(a)$ exists.

(iii) $\lim\limits_{x \to a} f(x) = f(a)$

f the above conditions fails, then the function is discontinu

Example – 1 : Test the continuity of the following function $f(x) = \begin{cases} \dfrac{x^2 - 4x + 3}{x - 1}, & x \neq 1 \\ -2 & x = 1 \end{cases}$ at $x = 1$

Sol^n : We have to test the continuity of the function at $x = 1$
Here $f(1) = -2$ (Given)

$$\text{Now } \lim_{x \to 1} \frac{x^2 - 4x + 3}{x - 1} = \lim_{x \to 1} \frac{(x-3)(x-1)}{x - 1} = \lim_{x \to 1}(x - 3) = 1 - 3 = -2$$

$$\text{and } \lim_{x \to 1} \frac{x^2 - 4x + 3}{x - 1} = \lim_{x \to 1}(x - 3)$$

$$= -2$$

$\therefore \lim f(x)$ exists and $= f(1)$
Thus the given function is continuous at $x = 1$.

xample – 2 : Show that $f(x) = \begin{cases} \dfrac{|x - a|}{x - a}, & x \neq a \\ 1 & x = a \end{cases}$ ***discontinuity at*** $x = a$.

l^n : $\text{L.H.L.} = \lim_{x \to a^-} f(x) = \lim_{x \to a^-} \frac{|x - a|}{x - a} = \lim_{h \to 0} \frac{|(a-h) - a|}{(a - h - a)} = \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1$

$\text{R.H.L.} = \lim_{x \to a^+} f(x)$

$$= \lim_{x \to a^+} \frac{|x - a|}{x - a} = \lim_{h \to 0} \frac{|(a + h) - a|}{a + h - a} = \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

$\therefore$ L.H.L. $\neq$ R.H.L.
Hence $f(x)$ is discontinuous at $x = a$.

Example – 3 : A function $f(x)$ is defined by $f(x) = \begin{cases} \cos x, \text{ when } x \geq 0 \\ -\cos x, \text{ when } x \leq 0 \end{cases}$

Examine the continuity of $f(x)$ at $x = 0$.

Sol^n : $f(0) = \cos 0 = 1$

$$\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-\cos x)$$

$$= \lim_{h \to 0} -\cos(0-h) = \lim_{h \to 0} -\cos h = -1$$

$$\text{R.H.L.} = \lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0^+} \cos x = \lim_{h \to 0} \cos(0+h) = \lim_{h \to 0} \cos h = 1$$

The given function $f(x)$ is continuous at $x = 0$

Example – 4 : **Show that $f(x) = [x]$ is not continuous at $x = n$, where n is an integer.**

Sol^n : We have, $f(n) = [n] = n$;

$$\lim_{x \to n+} = \lim_{h \to 0} f(n + h) = \lim_{h \to 0} [n + h] = n; \quad \{\because [n + h] = n\}$$

$$\lim_{x \to n-} = \lim_{h \to 0} f(n - h) = \lim_{h \to 0} [n - h] = (n - 1) \quad \{\because [n - h] = (n - 1)\}$$

Thus, $\lim_{x \to n+} f(x) \neq \lim_{x \to n-} f(x)$ and therefore, $\lim_{x \to n} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = n$.

Example – 5: **Show that the function $f(x) = \begin{cases} \frac{x}{|x|}, \text{ when } x \neq 0 \\ 1, \text{ when } x = 0 \end{cases}$ is discontinuous at $x = 0$**

Sol^n : It is being given that $f(0) = 1$

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} \frac{h}{|h|} = \lim_{h \to 0} \frac{h}{h} = 1$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} \frac{-h}{|h|} = \lim_{h \to 0} \frac{-h}{h} = -1$$

$$\therefore \lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$$

So, $\lim_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

ample – 6 : **Determine the value of k for which the function**

$$f(x) = \begin{cases} \frac{\sin 5x}{3x}, \text{ if } x \neq 0 \\ k, \text{ if } x = 0 \end{cases} \quad \textbf{is continuous at } x = 0$$

: Clearly, $f(0) = k$

Now, $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 5x}{3x} = \lim_{x \to 0} \left(\frac{\sin 5x}{5x} \times \frac{5}{3} \right) = \frac{5}{3} \cdot \lim_{x \to 0} \frac{\sin 5x}{5x} = \left(\frac{5}{3} \times 1 \right) = \frac{5}{3}$

For continuity of $f(x)$ at $x = 0$, we must have $f(0) = \lim_{x \to 0} f(x)$, i.e. $k = \frac{5}{3}$.

Hence, the required value of k is $\frac{5}{3}$.

Example – 12 : $f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}, \text{ if } x = 0 \\ 0, \text{ if } x = 0 \end{cases}$ at $x = 0$

Sol^{n} : $\lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1} = \dfrac{0 - 1}{0 + 1} = -1$

(As $x \to 0-$, $\dfrac{1}{x} \to -\infty$ and $e^{1/x} \to 0$)

Again $\lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0+} \dfrac{e^{1/x} - 1}{e^{1/x} + 1} = \lim\limits_{x \to 0+} \dfrac{1 - \dfrac{1}{e^{1/x}}}{1 + \dfrac{1}{e^{1/x}}} = \dfrac{1}{1} = 1$

($x \to 0+$, $\dfrac{1}{x} \to \infty \Rightarrow e^{1/x} \to \infty$)

Hence $\lim\limits_{x \to 0} f(x)$ doesnot exist at $x = 0$ and consequently $f(x)$ is not continuous at $x = 0$

Example – 13 : $f(x) = \begin{cases} 2x + 1, \text{ if } x \leq 0 \\ x, \quad \text{if } 0 < x \leq 1 \\ 2x - 1 \text{ if } x \geq 1 \end{cases}$ at $x = 0$

ol^{n} : Consider at $x = 0$

$\lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0-} (2x + 1) = \lim\limits_{x \to 0-h} [2(0 - h) + 1] = \lim\limits_{h \to 0} (-2h + 1) = 1$

$\lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0+} x = \lim\limits_{x \to 0+h} (0 + h) = \lim\limits_{h \to 0} h = 0$

Clearly $f(0) = 1$

$\therefore \lim\limits_{x \to 0} f(x)$ at $x = 0$ does not exist and consequently $f(x)$ is not continuous at $x = 0$

Consider at $x = 1$

$\lim\limits_{x \to 1-} f(x) = \lim\limits_{x \to 1-} x = \lim\limits_{x \to 1-h} (1 - h) = \lim\limits_{h \to 0} (1 - h) = 1$

$\lim\limits_{x \to 1+} f(x) = \lim\limits_{x \to 1+} (2x - 1) = \lim\limits_{x \to 0+h} [2(1 + h) - 1]$

$= \lim\limits_{h \to 0} (2 + 2h - 1) = \lim\limits_{h \to 0} (2h + 1) = 1$

$\therefore \lim\limits_{x \to 1} f(x) = 1$ And $f(x) = 2.\,1 - 1 = 2 - 1 = 1$

$\lim\limits_{x \to 1} f(x) = 1 = f(1)$, we conclude that the function is continuous at $x = 1$

3.3 SOME IMPORTANT DERIVATIVE USING FIRST PRINCIPLES

Theorem - 1. From first principles prove that $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is a fixed number
integer / rational.

Proof : Let $y = x^n$ .... (i)

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)^n$ .... (ii)

On subtracting (i) from (ii)

We get : $\delta y = (x + \delta x)^n - x^n$

$$\text{or } \frac{\delta y}{\delta x} = \frac{(x+\delta x)^n - x^n}{\delta x}$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{(x+\delta x)\to x} \frac{(x+\delta x)^n - x^n}{(x+\delta x) - x} \qquad [\because \delta x \to 0 \text{ means } x + \delta x \to x]$$

$$= nx^{n-1} \qquad \left[\because \lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$\boxed{\text{Thus } \frac{dy}{dx} = n\,x^{n-1} \text{ i.e., } \frac{d}{dx}(x^n) = n\,x^{n-1}}$$

amples : Find the derivative of the followings

(i) x^9, (ii) x^{-3}, (iii) $\sqrt[3]{x}$, (iv) $\frac{1}{\sqrt{x}}$

: We know that $\frac{d}{dx}(x^n) = n\,x^{n-1}$

So, we have :

(i) $\frac{d}{dx}(x^9) = 9 \,.\, x^{(9-1)} = 9x^8$

(ii) $\frac{d}{dx}(x^{-3}) = -3 \,.\, x^{(-3-1)} = -3x^{-4} = \frac{-3}{x^4}$

(iii) $\frac{d}{dx}(\sqrt[3]{x}) = \frac{d}{dx} x^{(1/3)} = \frac{1}{3}\, x^{\left(\frac{1}{3}-1\right)} = \frac{1}{3} x^{-2/3}$

(iv) $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) \Rightarrow \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}\; x^{\left(-\frac{1}{2}-1\right)} = -\frac{1}{2} x^{-3/2}.$

or $x^2 = 1$ or $x = \pm 1$

when $x = 1, y = 2 - 6 + 3 = -1$

and when $x = -1, y = -2 + 6 + 3 = 7$

∴ Points are $(1, -1)$ and $(-1, 7)$.

Example - 3. **At what point of the curve $y = x^2$ does the tangent to the curve make an angle of 45° with the axis of x ?**

Solⁿ : The equation of the curve is

$$y = x^2 \quad \therefore \quad \frac{dy}{dx} = 2x \qquad \text{.... (i)}$$

We are given that $\frac{dy}{dx} = \tan 45° = 1$ .... (ii)

From (i) and (ii),

$$2x = 1 \quad \text{or } x = \frac{1}{2}$$

$$\therefore \quad y = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

Thus the required point is $\left(\frac{1}{2}, \frac{1}{4}\right)$

Example -4. **Find the equation of the tangent and the normal to the curve $y = 2x^2 - 3x - 1$ at the point $(1, -2)$.**

Solⁿ : The equation of the curve is

$$y = 2x^2 - 3x - 1$$

$$\therefore \quad \frac{dy}{dx} = 4x - 3 \quad \therefore \quad \left[\frac{dy}{dx}\right]_{x=1} = 4 \,.\, 1 - 3 = 1$$

∴ Equation of the tangent at $(1, -2)$ is

$$y + 2 = 1\,(x - 1)$$

or $x - y - 3 = 0$

Equation of the normal is

$$y + 2 = -1(x - 1) \qquad \left[\text{Slope of the normal} = \frac{-1}{dy/dx}\right]$$

or $y + 2 = -x + 1$

∴ $x + y + 1 = 0$.

Example - 5. **Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to th y-coordinate.**

Solⁿ : The equation of the curve is

$$y = x^3 \qquad \text{.... (i)}$$

$$\therefore \quad \frac{dy}{dx} = 3x^2$$