depends upon the ...

Example – 1 : **Show that $\lim_{x\to 1}(3x+2)=5$.**

Solⁿ : Since, $(3x+2)-5=3x-3$,

$\therefore |(3x+2)-5|=3|x-1|$

Hence if $\in$ is given possible number, then

$|(3x+2)-5|<\in$ whenever $|x-1|<\in/3$

We can, therefore, choose $\delta=\frac{\in}{3}$, such that

$|(3x+2)-5|<\in$, whenever $0<|x-1|<\delta$.

Hence $\lim_{x\to 3}(3x+2)=5$

Example – 2 : **Show that $\lim_{x\to 3}\frac{x^2-9}{x-3}=6$**

Solⁿ : The function is not defined at $x=3$. For $x\neq 3$,

$\frac{x^2-9}{x-3}\equiv x+3$ and hence $\left|\frac{x^2-9}{x-3}-6\right|=|(x+3)-6|=|x-3|$.

Now $\left|\frac{x^2-9}{x-3}-6\right|<\in$, whenever $|x-3|<\in$.

Thus for given $\in>0$, we have found a positive number δ, which in this case, is equal to $\in$ itself such that

$0<|x-3|<\delta\Rightarrow\left|\frac{x^2-9}{x-3}-6\right|<\in$.

Hence $\lim_{x\to 3}\frac{x^2-9}{x-3}=6$

valuation of Left Hand and Right Hand limits :

The statements $x\to a^-$ means that 'x' is tending to a from the left hand side i.e. x is a number less
an a but very very close to 'a'. Therefore $x\to a^-$ is equivalent to $x=a-h$ where $h>0$ such that $h\to$
. Simillarly $x\to a^+$ is equivalent to $x=a+h$. Where $h\to 0+$. Thus, we have the following rules for
ding left hand and right hand limits at $x=a$.

ft Hand Limit :

To evaluate L.H.L of $f(x)$ at $x=a$ i.e.

$\lim_{x\to a^-}f(x)$ we proceed as follows :

Step 1 : Write $\lim_{x \to a^-} f(x)$

Step 2 : Put $x = a - h$ and replace $x \to a^-$ by

$h \to 0$ to obtain $\lim_{h \to 0} f(a - h)$

Step 3 : Simplify $\lim_{h \to 0} f(a - h)$ by using the formula for the given function.

Step 4 : The value obtain in step III is the LHL of $f(x)$ at $x = a$.

Right Hand Limit :

To evaluate RHL of $f(x)$ at $x = a$ i.e.

$\lim_{x \to a^+} f(x)$ we proceed as followes :

Step 1 : Write $\lim_{x \to a^+} f(x)$

Step 2 : Put $x = a + h$ and replace $x \to a^+$ by

$h \to 0$ to obtain $\lim_{h \to 0} f(a + h)$

Step 3 : Simplify $\lim_{h \to 0} f(a + h)$ by using the formula for the given function.

Step 4 : The value obtained in step III is the RHL of $f(x)$ at $x = a$.

Example – 1 : **Evaluate the left hand limit of the function.**

$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases} \quad \text{at } x = 4.$$

Solⁿ : (LHL of $f(x)$ at $x = 4$)

$$= \lim_{x \to 4^-} f(x) = \lim_{h \to 0+} f(4-h) = \lim_{h \to 0+} \frac{|4-h-4|}{4-h-4} = \lim_{h \to 0+} \frac{|-h|}{-h} = \lim_{h \to 0+} \frac{-h}{h} = \lim_{h \to 0+} (-1) = -1$$

Example - 2 : **Evaluate the right hand limit of the function.**

$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases} \quad \text{at } x = 4.$$

Solⁿ : (RHL of $f(x)$ at $x = 4$)

$$= \lim_{x \to 4^+} f(x) = \lim_{h \to 0+} f(4+h) = \lim_{h \to 0+} \frac{|4+h-4|}{4+h-4} = \lim_{h \to 0+} \frac{|h|}{h} = \lim_{h \to 0+} \frac{h}{h} = \lim_{h \to 0+} 1 = 1$$

Right hand limits of function defined by

So, $\lim_{x\to1} f(x)$

Example - 4 : If $f(x)=\begin{cases}\frac{x-|x|}{x}, & x\neq 0\\ 2 & x=0\end{cases}$ at x = 0, So that $\lim_{x\to0} f(x)$ does not exist.

Solⁿ : We have,

(LHL of $f(x)$ at x = 0)

$= \lim_{x\to0^-} f(x) = \lim_{h\to0+} f(0-h) = \lim_{h\to0+} \frac{-h-|-h|}{(-h)} = \lim_{h\to0+} \frac{-h-h}{-h}$

$= \lim_{h\to0+} \frac{-2h}{-h} = \lim_{h\to0+} 2 = 2.$ (RHL of $f(x)$ at x = 0)

$= \lim_{h\to0+} f(x) = \lim_{h\to0+} f(0+h) = \lim_{h\to0+} \frac{h-|h|}{h} = \lim_{h\to0+} \frac{h-h}{h}$

$= \lim_{h\to0} \frac{0}{h} = \lim_{h\to0} 0 = 0$

Since, $\lim_{x\to0^-} f(x) \neq \lim_{x\to0^+} f(x)$. So, $\lim_{x\to0} f(x)$ does not exist.

Example - 5 : If $f(x) = \begin{cases}5x-4, & 0<x\leq1\\ 4x^3-3x, & 1<x<2\end{cases}$,

Show that $\lim_{x\to1} f(x)$ exists.

Solⁿ : We have :

(LHL of $f(x)$ at x = 1)

$= \lim_{x\to1^-} f(x) = \lim_{h\to0+} f(1-h) = \lim_{h\to0+} 5(1-h)-4 = \lim_{h\to0+} 1-5h = 1$

(RHL of $f(x)$ at x = 1)

$= \lim_{x\to1^+} f(x) = \lim_{h\to0+} f(1+h) = \lim_{h\to0+} 4(1+h)^3 - 3(1+h)$

$= 4(1)^3 - 3(1) = 1$

Clearly, $\lim_{x\to1-} f(x) = \lim_{x\to1+} f(x)$. So, $\lim_{x\to1} f(x)$ exists and is equal to 1.

Hence $a = 1$ and $b = 2 - 3$

Expansion Formulaes :

Sometimes following expansion are very useful to evaluate limits.

(a) $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$

(b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^2}{3!} + ...$

(c) $a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + ...$

(d) $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$

(e) $\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...$

(f) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + ...$

(g) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...$

(h) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...$

(i) $\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}x^5 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\frac{x^7}{7}$

(j) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + ...$

(k) $\sec x = 1 + \frac{x^2}{2!} + 5\frac{x^4}{4!} + ...$

Theorem -2 : **Prove that** $\lim_{x\to 0}\left(\frac{a^x-1}{x}\right) = \log_e a, \ (a > 0)$

Proof : $\lim_{x\to 0}\frac{a^x-1}{x}$

putting $a^x - 1 = y$, $a^x = y + 1$, $\log a^x = \log(y+1)$, $x \log a = \log(y+1)$

$x = \frac{\log(1+y)}{\log a}$ [When $x \to 0$ i.e. $y \to 0$]

$$= \lim_{y\to 0}\frac{y}{\frac{\log(1+y)}{\log a}} = \lim_{y\to 0}\frac{\log a}{\frac{\log(1+y)}{y}} = \frac{\log a}{\lim_{y\to 0}\frac{\log(1+y)}{y}} = \log_e a \quad \left[\because \lim_{y\to 0}\frac{\log(1+y)}{y} = 1\right]$$

Another Method :

$$\lim_{x\to 0}\left(\frac{a^x-1}{x}\right) = \lim_{x\to 0}\left\{\frac{\{1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + ...\} - 1}{x}\right\} \quad \text{[on expending } a^x\text{]}$$

$$= \lim_{x\to 0}\frac{x\left\{(\log a) + \frac{x}{2!}(\log a)^2 + ...\right\}}{x} = \lim_{x\to 0}\left\{\log a + \frac{x}{2!}(\log a)^2 + ...\right\}$$

$= \log a$ [Putting $x = 0$]

$$\therefore \lim_{x\to 0}\left(\frac{a^x-1}{x}\right) = \log_e a$$

Theorem - 3 : **Prove that** $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right) = 1$

Proof : $$\lim_{x\to 0}\left(\frac{e^x-1}{x}\right) = \lim_{x\to 0}\left\{\frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right) - 1}{x}\right\} \quad \text{[on expanding } e^x\text{]}$$

$$= \lim_{x\to 0}\left\{\frac{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)}{x}\right\} = \lim_{x\to 0}\left\{\frac{x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + ...\right)}{x}\right\}$$

$$= \lim_{x\to 0}\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + ...\right) \quad \therefore \lim_{x\to 0}\left(\frac{e^x-1}{x}\right) = 1$$

Example-1. Evaluate : (i) $\lim_{x\to 0}\frac{\sin 2x}{x}$ (ii) $\lim_{x\to 0}\frac{\sin 3x}{5x}$ (iii) $\lim_{x\to 0}\left(\frac{\sin ax}{\sin bx}\right)$

Solⁿ : (i) $\lim_{x\to 0}\frac{\sin 2x}{x} = \lim_{x\to 0}\left(2\times\frac{\sin 2x}{2x}\right) = 2\,.\lim_{2x\to 0}\frac{\sin 2x}{2x}$

[∵ when $x \to 0$, then $2x \to 0$]

$= (2 \times 1) = 1$ $\left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right]$

(ii) $\lim_{x\to 0}\frac{\sin 3x}{5x} = \lim_{x\to 0}\left(\frac{3}{5}\cdot\frac{\sin 3x}{3x}\right) = \frac{3}{5}\cdot\lim_{3x\to 0}\frac{\sin 3x}{3x}$

[∵ when $x \to 0$, then $3x \to 0$]

$= \left(\frac{3}{5}\times 1\right) = \frac{3}{5}$ $\left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right]$

(iii) $\lim_{x\to 0}\left(\frac{\sin ax}{\sin bx}\right) = \lim_{x\to 0}\frac{ax\,.\left(\frac{\sin ax}{ax}\right)}{bx\,.\left(\frac{\sin bx}{bx}\right)}$

$= \frac{a}{b}\cdot\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)}{\left(\frac{\sin bx}{bx}\right)} = \frac{a}{b}\cdot\frac{\lim_{ax\to 0}\frac{\sin ax}{ax}}{\lim_{bx\to 0}\frac{\sin bx}{bx}} = \left(\frac{a}{b}\times\frac{1}{1}\right) = \frac{a}{b}$

[∵ $x \to 0$, ⇒ $ax \to 0$ and $bx \to 0$]

$= \left(\frac{a}{b}\times\frac{1}{1}\right) = \frac{a}{b}$ $\left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right]$

Example–2 : **Evaluate :** $\lim_{x\to 0}\frac{\sin 5x}{\tan 3x}$

Solⁿ : $\lim_{x\to 0}\frac{\sin 5x}{\tan 3x} = \lim_{x\to 0}\frac{5x\,.\left(\frac{\sin 5x}{5x}\right)}{3x\,.\left(\frac{\tan 3x}{3x}\right)} = \frac{5}{3}\cdot\frac{\lim_{5x\to 0}\left(\frac{\sin 5x}{5x}\right)}{\lim_{3x\to 0}\left(\frac{\tan 3x}{3x}\right)} = \left(\frac{5}{3}\times\frac{1}{1}\right) = \frac{5}{3}$

Example–3 : **Evaluate :** $\lim_{x\to 0}\frac{\sin x^\circ}{x}$

Solⁿ : We know that $x^\circ = \left(\frac{\pi x}{180}\right)^c$

$\therefore \lim_{x\to 0}\frac{\sin x^\circ}{x} = \lim_{x\to 0}\frac{\sin\left(\frac{\pi x}{180}\right)}{x} = \lim_{x\to 0}\left\{\frac{\sin\left(\frac{\pi x}{180}\right)}{\left(\frac{\pi x}{180}\right)}\times\frac{\pi}{180}\right\}$

$= \frac{\pi}{180}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$, (where $\theta = \frac{\pi x}{180}$) [clearly, $x \to 0 \Rightarrow \frac{\pi x}{180} \to 0 \Rightarrow \theta \to 0$]

$= \left(\frac{\pi}{180}\times 1\right) = \frac{\pi}{180}$.

$\quad y \to 0 \qquad \partial y$

Example – 14 : Examine the existence of the following limits.

(a) $\lim_{x \to 0} \frac{|x|}{x}$ (b) $\lim_{x \to \infty} \frac{x}{|x|}$ (c) $\lim_{x \to \infty} \frac{x^2 - x}{[x^2 - x]}$

Solⁿ : (a) $\lim_{x \to 0-} \frac{|x|}{x} = \lim_{x \to 0-} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1$

$\lim_{x \to 0+} \frac{|x|}{x} = \lim_{x \to 0+} \frac{|+h|}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$

Hence $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

(b) $\lim_{x \to \infty} \frac{x}{[x]}$

We know $\forall x \in R \qquad [x] \le x < [x] + 1$

$\Rightarrow \frac{[x]}{[x]} \le \frac{x}{[x]} < 1 + \frac{1}{[x]} = 1 \le \lim_{x \to \infty} \frac{x}{[x]} \le \lim_{x \to \infty} 1 + \frac{1}{[x]}$

$= 1 \le \lim_{x \to \infty} \frac{x}{[x]} \le 1 \qquad$ [Hence $\lim_{x \to \infty} \frac{x}{[x]} = 1$]

(c) $\lim_{x \to \infty} \frac{x^2 - x}{[x^2 - x]}$ set $y = x^2 - x$ as $x \to \infty$, i.e. $y \to \infty$

$= \lim_{y \to \infty} \frac{y}{[y]}$ exist's and equal to 1

Definition : A function $f(x)$ is said to be continuous at $x = a$ if :

(i) $\lim_{x \to a} f(x)$ exists,

i.e. right hand and left hand limits exist and are equal.

(ii) $f(a)$ exists.

(iii) $\lim_{x \to a} f(x) = f(a)$

f the above conditions fails, then the function is discontinu

Example – 1 : Test the continuity of the following function $f(x) = \begin{cases} \dfrac{x^2 - 4x + 3}{x - 1}, & x \neq 1 \\ -2, & x = 1 \end{cases}$ at $x = 1$

Solⁿ : We have to test the continuity of the function at $x = 1$

Here $f(1) = -2$ (Given)

Now $\lim\limits_{x \to 1} \dfrac{x^2 - 4x + 3}{x - 1} = \lim\limits_{x \to 1} \dfrac{(x-3)(x-1)}{x-1} = \lim\limits_{x \to 1}(x-3) = 1 - 3 = -2$

and $\lim\limits_{x \to 1} \dfrac{x^2 - 4x + 3}{x - 1} = \lim\limits_{x \to 1}(x-3)$

$= -2$

$\therefore \lim f(x)$ exists and $= f(1)$

Thus the given function is continuous at $x = 1$.

xample – 2 : Show that $f(x) = \begin{cases} \dfrac{|x-a|}{x-a}, & x \neq a \\ 1 & x = a \end{cases}$ *discontinuity at* $x = a$.

l : L.H.L. $= \lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^-} \dfrac{|x-a|}{x-a} = \lim\limits_{h \to 0} \dfrac{|(a-h)-a|}{(a-h-a)} = \lim\limits_{h \to 0} \dfrac{|-h|}{-h} = \lim\limits_{h \to 0} \dfrac{h}{-h} = -1$

R.H.L. $= \lim\limits_{x \to a^+} f(x)$

$= \lim\limits_{x \to a^+} \dfrac{|x-a|}{x-a} = \lim\limits_{h \to 0} \dfrac{|(a+h)-a|}{a+h-a} = \lim\limits_{h \to 0} \dfrac{|h|}{h} = \lim\limits_{h \to 0} \dfrac{h}{h} = 1$

$\therefore$ L.H.L. $\neq$ R.H.L.

Hence $f(x)$ is discontinuous at $x = a$.

Example – 3 : A function $f(x)$ is defined by $f(x) = \begin{cases} \cos x, \text{ when } x \geq 0 \\ -\cos x, \text{ when } x \leq 0 \end{cases}$

Examine the continuity of $f(x)$ at $x = 0$.

Solⁿ : $f(0) = \cos 0 = 1$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-\cos x)$$

$$= \lim_{h\to 0} -\cos(0-h) = \lim_{h\to 0} -\cos h = -1$$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x)$$

$$= \lim_{x\to 0^+} \cos x = \lim_{h\to 0} \cos(0+h) = \lim_{h\to 0} \cos h = 1$$

The given function $f(x)$ is continuous at $x = 0$

Example – 4 : **Show that $f(x) = [x]$ is not continuous at $x = n$, where n is an integer.**

Solⁿ : We have, $f(n) = [n] = n$;

$$\lim_{x\to n+} = \lim_{h\to 0} f(n+h) = \lim_{h\to 0} [n+h] = n; \quad \{\because [n+h] = n\}$$

$$\lim_{x\to n-} = \lim_{h\to 0} f(n-h) = \lim_{h\to 0} [n-h] = (n-1) \quad \{\because [n-h] = (n-1)\}$$

Thus, $\lim_{x\to n+} f(x) \neq \lim_{x\to n-} f(x)$ and therefore, $\lim_{x\to n} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = n$.

Example – 5: **Show that the function** $f(x) = \begin{cases} \dfrac{x}{|x|}, \text{ when } x \neq 0 \\ 1, \text{ when } x = 0 \end{cases}$ **is discontinuous at $x = 0$**

Solⁿ : It is being given that $f(0) = 1$

$$\lim_{x\to 0+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{h}{|h|} = \lim_{h\to 0} \frac{h}{h} = 1$$

$$\lim_{x\to 0-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{-h}{|h|} = \lim_{h\to 0} \frac{-h}{h} = -1$$

$$\therefore \lim_{x\to 0+} f(x) \neq \lim_{x\to 0-} f(x)$$

So, $\lim_{x\to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

ample – 6 : **Determine the value of k for which the function**

$$f(x) = \begin{cases} \dfrac{\sin 5x}{3x}, \text{ if } x \neq 0 \\ k, \text{ if } x = 0 \end{cases} \text{ is continuous at } x = 0$$

: Clearly, $f(0) = k$

Now, $\lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{\sin 5x}{3x} = \lim_{x\to 0} \left(\dfrac{\sin 5x}{5x} \times \dfrac{5}{3}\right) = \dfrac{5}{3} \cdot \lim_{x\to 0} \dfrac{\sin 5x}{5x} = \left(\dfrac{5}{3} \times 1\right) = \dfrac{5}{3}$

For continuity of $f(x)$ at $x = 0$, we must have $f(0) = \lim_{x\to 0} f(x)$, i.e. $k = \dfrac{5}{3}$.

Hence, the required value of k is $\dfrac{5}{3}$.

Example – 12 : $f(x) = \begin{cases} \frac{e^{1/x} - 1}{e^{1/x} + 1}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$

Solⁿ : $\lim_{x \to 0-} f(x) = \lim_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x} + 1} = \frac{0 - 1}{0 + 1} = -1$

(As $x \to 0-$, $\frac{1}{x} \to -\infty$ and $e^{1/x} \to 0$)

Again $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{e^{1/x} - 1}{e^{1/x} + 1} = \lim_{x \to 0+} \frac{1 - \frac{1}{e^{1/x}}}{1 + \frac{1}{e^{1/x}}} = \frac{1}{1} = 1$

($x \to 0+$, $\frac{1}{x} \to \infty \Rightarrow e^{1/x} \to \infty$)

Hence $\lim_{x \to 0} f(x)$ doesnot exist at $x = 0$ and consequently $f(x)$ is not continuous at $x = 0$

Example – 13 : $f(x) = \begin{cases} 2x + 1, & \text{if } x \leq 0 \\ x, & \text{if } 0 < x \leq 1 \\ 2x - 1 & \text{if } x \geq 1 \end{cases}$ at $x = 0$

Solⁿ : Consider at $x = 0$

$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (2x + 1) = \lim_{x \to 0-h} [2(0 - h) + 1] = \lim_{h \to 0} (-2h + 1) = 1$

$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x = \lim_{x \to 0+h} (0 + h) = \lim_{h \to 0} h = 0$

Clearly $f(0) = 1$

$\therefore \lim_{x \to 0} f(x)$ at $x = 0$ does not exist and consequently $f(x)$ is not continuous at $x = 0$

Consider at $x = 1$

$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} x = \lim_{x \to 1-h} (1 - h) = \lim_{h \to 0} (1 - h) = 1$

$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (2x - 1) = \lim_{x \to 0+h} [2(1 + h) - 1]$

$= \lim_{h \to 0} (2 + 2h - 1) = \lim_{h \to 0} (2h + 1) = 1$

$\therefore \lim_{x \to 1} f(x) = 1$ And $f(x) = 2.1 - 1 = 2 - 1 = 1$

$\lim_{x \to 1} f(x) = 1 = f(1)$, we conclude that the function is continuous at $x = 1$

3.3 SOME IMPORTANT DERIVATIVE USING FIRST PRINCIPLES

Theorem - 1. From first principles prove that $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is a fixed number [illegible] integer / rational.

Proof : Let $y = x^n$ .... (i)

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)^n$ .... (ii)

On subtracting (i) from (ii)

We get : $\delta y = (x + \delta x)^n - x^n$

$$\text{or } \frac{\delta y}{\delta x} = \frac{(x+\delta x)^n - x^n}{\delta x}$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{(x+\delta x)\to x} \frac{(x+\delta x)^n - x^n}{(x+\delta x) - x} \qquad [\because \delta x \to 0 \text{ means } x + \delta x \to x]$$

$$= nx^{n-1} \qquad \left[\because \lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$\boxed{\text{Thus } \frac{dy}{dx} = n\,x^{n-1} \text{ i.e., } \frac{d}{dx}(x^n) = n\,x^{n-1}}$$

Examples : **Find the derivative of the followings**

(i) x^9, (ii) x^{-3}, (iii) $\sqrt[3]{x}$, (iv) $\frac{1}{\sqrt{x}}$

: We know that $\frac{d}{dx}(x^n) = n\,x^{n-1}$

So, we have :

(i) $\frac{d}{dx}(x^9) = 9 \,.\, x^{(9-1)} = 9x^8$

(ii) $\frac{d}{dx}(x^{-3}) = -3 \,.\, x^{(-3-1)} = -3x^{-4} = \frac{-3}{x^4}$

(iii) $\frac{d}{dx}(\sqrt[3]{x}) = \frac{d}{dx} x^{(1/3)} = \frac{1}{3}\, x^{\left(\frac{1}{3}-1\right)} = \frac{1}{3} x^{-2/3}$

(iv) $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}\, x^{\left(-\frac{1}{2}-1\right)} = -\frac{1}{2} x^{-3/2}$.

or $x^2 = 1$ or $x = \pm 1$

when $x = 1, y = 2 - 6 + 3 = -1$

and when $x = -1, y = -2 + 6 + 3 = 7$

∴ Points are $(1, -1)$ and $(-1, 7)$.

Example - 3. **At what point of the curve $y = x^2$ does the tangent to the curve make an angle of 45° with the axis of x ?**

Solⁿ : The equation of the curve is

$$y = x^2 \quad \therefore \quad \frac{dy}{dx} = 2x \qquad \text{.... (i)}$$

We are given that $\frac{dy}{dx} = \tan 45° = 1$ .... (ii)

From (i) and (ii),

$$2x = 1 \quad \text{or } x = \frac{1}{2}$$

$$\therefore \quad y = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

Thus the required point is $\left(\frac{1}{2}, \frac{1}{4}\right)$

Example -4. **Find the equation of the tangent and the normal to the curve $y = 2x^2 - 3x - 1$ at the point $(1, -2)$.**

Solⁿ : The equation of the curve is

$$y = 2x^2 - 3x - 1$$

$$\therefore \quad \frac{dy}{dx} = 4x - 3 \;\therefore\; \left[\frac{dy}{dx}\right]_{x=1} = 4 \,.\, 1 - 3 = 1$$

∴ Equation of the tangent at $(1, -2)$ is

$$y + 2 = 1\,(x - 1)$$

or $x - y - 3 = 0$

Equation of the normal is

$$y + 2 = -1(x - 1) \qquad \left[\text{Slope of the normal} = \frac{-1}{dy/dx}\right]$$

or $y + 2 = -x + 1$

∴ $x + y + 1 = 0$.

Example - 5. **Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to th y-coordinate.**

Solⁿ : The equation of the curve is

$$y = x^3 \qquad \text{.... (i)}$$

$$\therefore \quad \frac{dy}{dx} = 3x^2$$